Drawing Curves with Epicycles from Discrete Points

William F. Barnes

Abstract

An ordered list of points in the plane can be converted into a periodic curve drawn by a chain of rotating circles. The conversion is the discrete Fourier transform (DFT): each Fourier coefficient supplies the radius and initial phase of one epicycle, while its Fourier index supplies the rotation frequency. This article develops the construction directly from Cartesian point data. It also explains equal-arclength resampling, signed frequencies, exact interpolation at the sampled points, lower-resolution approximations, and the special treatment required when an open stroke is made cyclic by returning from its last point to its first.

1 The problem

Suppose a curve is supplied not by an equation but by an ordered list of Cartesian locations

$$\mathcal{P} = \big((x_0, y_0), (x_1, y_1), \ldots, (x_{P-1}, y_{P-1})\big). \tag{1}$$

The points may have been typed, measured, or recorded while a pointer was dragged across a screen. Their order matters: it specifies the order in which the drawing tip is intended to visit them.

The goal is to construct a chain of rotating vectors whose final tip follows a continuous curve through the data. In Cartesian form, a chain of M epicycles has tip

$$x(t) = \sum_{m=0}^{M-1} r_m \cos\big(2\pi\nu_m t + \phi_m\big), \tag{2}$$

$$y(t) = \sum_{m=0}^{M-1} r_m \sin\big(2\pi\nu_m t + \phi_m\big), \tag{3}$$

where $0 \le t < 1$. The quantities r_m, ν_m, and ϕ_m are the radius, signed frequency, and initial phase of the mth rotating vector. We must determine these quantities from the point list in Equation (1).

2 Preparing the point data

2.1 Why resampling is useful

Pointer events and measured locations are rarely spaced uniformly. A slowly drawn portion of a stroke may contain many points, while a quickly drawn portion may contain only a few. If the raw list is sent directly to the DFT, equal increments of the curve parameter correspond to equal increments of the list index, not equal distances along the drawing. The resulting epicycle tip therefore moves slowly through densely sampled regions and quickly through sparse ones.

A cleaner geometric representation is obtained by resampling the ordered path at approximately equal arclength. Let the successive segment lengths be

$$d_q = \sqrt{(x_{q+1} - x_q)^2 + (y_{q+1} - y_q)^2}. \tag{4}$$

For a cyclic curve, indices are interpreted modulo P, so the final segment has length

$$d_{P-1} = \sqrt{(x_0 - x_{P-1})^2 + (y_0 - y_{P-1})^2}. \tag{5}$$

The total polygonal length is

$$L = \sum_{q=0}^{P-1} d_q. \tag{6}$$

Choose a desired sample count N and interpolate along the polygonal path at the distances

$$s_p = \frac{pL}{N}, \qquad p = 0, 1, \ldots, N-1. \tag{7}$$

This produces a new ordered list of N nearly equally spaced locations. The first point is included, but it is not repeated at the end; periodicity already identifies the parameter values $t = 0$ and $t = 1$.

If the supplied points are already suitable parameter samples, the resampling step may be omitted and one may simply take $N = P$.

2.2 Open strokes and the return interval

Every finite sum in Equations (2)–(3) is periodic. Consequently, an epicycle tip must eventually return to its initial position. When the user's final point differs from the first, a cyclic parameterization must contain a return interval from (x_{P-1}, y_{P-1}) to (x_0, y_0).

The return interval may be used for the Fourier calculation without being displayed as part of the intended stroke. If

$$L_{\text{open}} = \sum_{q=0}^{P-2} d_q, \tag{8}$$

then the visible fraction of one cycle is

$$\tau = \frac{L_{\text{open}}}{L_{\text{open}} + d_{P-1}}. \tag{9}$$

The epicycle chain is evaluated for the full interval $0 \le t < 1$, preserving cyclic motion, while ink may be deposited only for $0 \le t < \tau$. Thus the tip traverses the mathematical closing portion but the straight connection between the user's endpoints need not appear in the drawing.

3 Complex form of the sampled curve

Represent the pth resampled point by one complex number,

$$z_p = x_p + \mathrm{i}y_p, \qquad p = 0, 1, \ldots, N-1. \tag{10}$$

Also assign the uniformly spaced parameter values

$$t_p = \frac{p}{N}. \tag{11}$$

The inverse discrete Fourier representation of the data is

$$z_p = \sum_{j=0}^{N-1} Q_j \mathrm{e}^{\mathrm{i}2\pi jp/N}, \tag{12}$$

where each $Q_j \in \mathbb{C}$ is constant. The complex exponential represents a rotating vector, and Q_j sets its size and initial direction.

Equation (12) already has the structure of an epicycle chain. The remaining task is to recover the coefficients Q_j from the samples z_p.

4 The discrete Fourier transform

Multiply Equation (12) by $\mathrm{e}^{-\mathrm{i}2\pi kp/N}$ and sum over all sample indices p:

$$\sum_{p=0}^{N-1} z_p \mathrm{e}^{-\mathrm{i}2\pi kp/N} = \sum_{p=0}^{N-1}\sum_{j=0}^{N-1} Q_j \mathrm{e}^{\mathrm{i}2\pi(j-k)p/N} \tag{13}$$

$$= \sum_{j=0}^{N-1} Q_j \sum_{p=0}^{N-1} \mathrm{e}^{\mathrm{i}2\pi(j-k)p/N}. \tag{14}$$

For $j, k \in \{0, 1, \ldots, N-1\}$, the roots-of-unity identity is

$$\frac{1}{N}\sum_{p=0}^{N-1} \mathrm{e}^{\mathrm{i}2\pi(j-k)p/N} = \delta_{jk}, \tag{15}$$

where δ_{jk} is the Kronecker delta. Equation (14) therefore reduces to NQ_k, giving the DFT

$$\boxed{Q_k = \frac{1}{N}\sum_{p=0}^{N-1} z_p \mathrm{e}^{-\mathrm{i}2\pi kp/N}} \qquad k = 0, 1, \ldots, N-1. \tag{16}$$

4.1 Why the orthogonality identity works

If $j = k$, every term in Equation (15) equals one, so the sum is N. If $j \neq k$, set

$$w = \mathrm{e}^{\mathrm{i}2\pi(j-k)/N}. \tag{17}$$

Here $w \neq 1$ but $w^N = 1$. The finite geometric series gives

$$\sum_{p=0}^{N-1} w^p = \frac{1-w^N}{1-w} = 0. \tag{18}$$

After division by N, the two cases are exactly those of the Kronecker delta.

4.2 Exact recovery at the samples

Substitution of Equation (16) into Equation (12) gives

$$\sum_{j=0}^{N-1} Q_j \mathrm{e}^{\mathrm{i}2\pi jp/N} = \sum_{p'=0}^{N-1} z_{p'} \left(\frac{1}{N}\sum_{j=0}^{N-1} \mathrm{e}^{\mathrm{i}2\pi j(p-p')/N} \right) \tag{19}$$

$$= \sum_{p'=0}^{N-1} z_{p'}\delta_{pp'} = z_p. \tag{20}$$

Thus the full N-term reconstruction passes exactly through all N samples, apart from numerical roundoff. Between the samples it follows a trigonometric interpolant, not the polygonal connect-the-dots path.

5 From Fourier coefficients to epicycles

5.1 Signed frequencies

The indices $0, 1, \ldots, N-1$ are sufficient for the discrete inverse in Equation (12). For a continuous epicycle animation, however, a balanced collection of positive and negative frequencies is more natural. Define

$$\nu_j = \begin{cases} j, & 0 \le j \le \lfloor N/2 \rfloor, \\ j-N, & \lfloor N/2 \rfloor < j \le N-1. \end{cases} \tag{21}$$

For even N, the Nyquist index $N/2$ has been assigned the positive sign; the opposite convention is equally valid.

At a sample time $t_p = p/N$,

$$\mathrm{e}^{\mathrm{i}2\pi(j-N)t_p} = \mathrm{e}^{\mathrm{i}2\pi jp/N}\mathrm{e}^{-\mathrm{i}2\pi p} = \mathrm{e}^{\mathrm{i}2\pi jp/N}. \tag{22}$$

Replacing j by ν_j therefore leaves every sampled point unchanged while producing the desired clockwise and counterclockwise rotations between sample times.

5.2 Radius and phase

Write each coefficient in polar form,

$$Q_j = r_j \mathrm{e}^{\mathrm{i}\phi_j}, \qquad r_j = |Q_j|, \qquad \phi_j = \mathrm{Arg}(Q_j). \tag{23}$$

The continuous Fourier curve is then

$$\boxed{Z(t) = \sum_{j=0}^{N-1} r_j \mathrm{e}^{\mathrm{i}(2\pi\nu_j t + \phi_j)}} \qquad 0 \le t < 1. \tag{24}$$

Taking real and imaginary parts of Equation (24) recovers Equations (2)–(3). Each summand is one rotating vector. A negative ν_j rotates clockwise, a positive ν_j rotates counterclockwise, and $\nu_0 = 0$ supplies a stationary translation of the entire curve.

5.3 The geometric chain

The rotating vectors may be arranged in any order without changing their final sum. Let $k_0, k_1, \ldots, k_{M-1}$ be the chosen order of the retained coefficients. Define the successive centers by

$$C_0(t) = 0, \tag{25}$$

$$C_{m+1}(t) = C_m(t) + Q_{k_m}\mathrm{e}^{\mathrm{i}2\pi\nu_{k_m}t}, \qquad m = 0, 1, \ldots, M-1. \tag{26}$$

The mth circle is centered at $C_m(t)$ and has radius $|Q_{k_m}|$. Its radius vector ends at $C_{m+1}(t)$. The final drawing tip is

$$C_M(t) = \sum_{m=0}^{M-1} Q_{k_m}\mathrm{e}^{\mathrm{i}2\pi\nu_{k_m}t}. \tag{27}$$

With all N coefficients retained, this is Equation (24).

For display, it is often useful to order coefficients by decreasing radius. The largest circles then establish the broad shape and the smaller circles add progressively finer detail. This reordering changes the intermediate circle centers but not the position of the final tip.

6 Resolution and approximation

Using all N coefficients reproduces every sampled point exactly. A simpler chain may be obtained by retaining only $M < N$ coefficients. If $\mathcal{K}_M$ is the selected index set, the approximation is

$$Z_M(t) = \sum_{j \in \mathcal{K}_M} Q_j \mathrm{e}^{\mathrm{i}2\pi\nu_j t}. \tag{28}$$

Two common selection rules are:

1. retain a symmetric low-frequency band around zero; or
2. retain the M coefficients having the largest radii $|Q_j|$.

The first rule behaves like ordinary Fourier low-pass filtering. The second usually gives a visually efficient epicycle chain because every retained circle makes a comparatively large contribution.

Sharp corners require many frequencies. Truncation rounds the corners and may introduce oscillation near abrupt changes, the familiar Gibbs phenomenon. Smooth curves are generally reproduced well with far fewer epicycles.

7 Complete algorithm

The entire conversion from point data to epicycles can be stated compactly.

Listing 1: Point data to epicycle coefficients.

```
INPUT: ordered Cartesian points raw[0 ... P-1]
       desired sample count N

1. Treat raw as an ordered polygonal path.
2. Include a final return segment raw[P-1] -> raw[0].
3. Resample that cyclic path at N equal-arclength positions,
   producing sample[p] = (x[p], y[p]).
4. Set z[p] = x[p] + i*y[p].
5. For each j = 0 ... N-1:
       Q[j] = 0
       For each p = 0 ... N-1:
           Q[j] += z[p] * exp(-i*2*pi*j*p/N)
       Q[j] /= N

       If j <= floor(N/2): frequency[j] = j
       Else:               frequency[j] = j - N

       radius[j] = abs(Q[j])
       phase[j]  = arg(Q[j])
6. Optionally sort the records by decreasing radius.
OUTPUT: (Q[j], frequency[j], radius[j], phase[j])
```

At animation time, the chain is evaluated as follows.

Listing 2: Animating and drawing the epicycle chain.

```
INPUT: retained coefficient records in display order
       cycle parameter t in [0,1)
```

```
center = 0 + 0*i
For each coefficient c:
    vector = c.Q * exp(i*2*pi*c.frequency*t)
    draw circle centered at center with radius c.radius
    draw radius vector from center to center + vector
    center += vector
tip = center

If t lies in the visible portion of the cycle:
    append tip to the ink trace
Else:
    move the tip without depositing ink
```

The direct DFT requires $O(N^2)$ arithmetic operations. This is entirely adequate for drawings containing a few hundred samples and has the advantage of transparent implementation. For much larger data sets, the same coefficients can be computed in $O(N \log N)$ operations by a fast Fourier transform.

8 Practical details

- **Do not duplicate the first sample.** Store N samples on $0 \leq t < 1$. Repeating the initial point as sample N gives it extra weight.
- **Remove consecutive duplicates.** Zero-length polygonal segments complicate arclength interpolation but carry no geometric information.
- **Choose a coordinate convention.** Mathematical coordinates use an upward-positive y-axis, whereas screen coordinates usually increase downward. Negate the screen y-coordinate before the DFT, or negate it consistently during rendering.
- **Centering is optional.** Subtracting the sample centroid before the DFT makes $Q_0 = 0$. Keeping the original coordinates instead stores the translation in the zero-frequency coefficient.
- **Scale only for display.** Multiplying all input coordinates by a common factor multiplies every radius by that factor but does not change frequencies or phases.
- **Keep sampling order.** The DFT reconstructs the ordered traversal, not merely the unordered set of locations.
- **Distinguish the guide from the ink.** The user's open stroke, the Fourier preview, the moving circles, and the deposited trace are separate visual objects. In particular, the closing interval needed for periodicity can remain invisible even while the epicycle tip traverses it.

9 Conclusion

An ordered list of Cartesian locations contains enough information to build a complete epicycle drawing. Equal-arclength resampling converts the raw path into uniform parameter data. The DFT then produces complex coefficients whose magnitudes, arguments, and signed indices are respectively the radii, initial phases, and rotation frequencies of the circles. Their vector sum gives a continuous periodic curve, and the full coefficient set interpolates every sample exactly. Truncating the set replaces exactness with a smaller, visually simpler chain. The result is a direct route from discrete point data to a cyclic curve drawn by epicycles.